Certain Problems in the Application of Mathematical Economics*

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This paper will divide the problems in the application of mathematical economics into two arbitrary categories: (1) we will discuss the problems concerning the meaning and accuracy of the empirical information with which economists must deal; and (2) we will present some of the theoretical problems inherent in a view of economic mechanisms as sets of interdependent stochastic relationships.

When we investigate the first problem area, we find that the idea of operationalism which we might adopt from engineering and the physical sciences can be applied only with some difficulty. In many experimental situations in engineering and experimental science, the operational significance of a measurement procedure can be fairly simply described and incorporated into the conclusions which may validly be drawn from the experiment. However, in economics we find that our empirical evidence comes about in some very complex and indirect ways. A full assessment of the operational significance of procedures as complex as the national census requires a level of knowledge and analysis which has simply not been attained in economic science.

Regarding the second problem category, we find that new techniques are required to investigate satisfactorily many of our hypotheses about economic reality. As soon as we admit the factors of interdependence and growth (both in a physical sense and in an organizational and intellectual sense), we find that much of classical statistical methodology is inappropriate for our problems. If we add to our difficulties the very reasonable hypothesis that many aspects of the economic system may purposefully attempt to hide information from statistical discovery, then we are faced with problems which are of the greatest importance but are also far from resolution.

In conclusion, we simply point out that, since the problems of applied mathematical economics are so severe, the scope for solutions of major importance is immense.

1. Introduction

There is a certain amount of opinion both within and outside economics which holds that economics is, or ought to be, a nonmathematical if not nonscientific subject. While I would not agree with this view categorically, some of the arguments in its support cannot be lightly dismissed. The strongest argument that the nonmathematical school can put forward is that the uncritical application of mathematical techniques which were developed for, and found useful in, other sciences will probably not be

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successful in economics. This argument is supported on the basis of the unique nature of economic problems. This and other arguments pointing to the difficulties attendant upon the mathematization of economics have been discussed to some extent in the literature. Of particular interest are the works of Oskar Morgenstern[1], [2].

In this paper I do not want to pursue further the general question of the role of mathematics in economics. Rather, I will discuss one particular problem in economics and attempt to indicate that the application of mathematical methods which fall generally in the category of system theory, while at first sight seemingly reasonable, is in fact misleading and inappropriate to what one believes to be economic reality. However, I will go on to argue that the incorporation of some of the ideas of system theory into a mathematical model which seems to better characterize the economic problem leads to interesting results.

For the purpose of this discussion I will subdivide mathematical methods in economics into two broad categories. The first category consists of a large class of models which reduce mathematically to sets of simultaneous differential or difference equations. The development of these models was originally stimulated by the great success of classical mechanics. The formulations are formally quite similar to those of mechanics. The first complete specification of a model of this kind, applied to an entire economy, was given in 1874 by L. Walras[3]. By about 1930, the problem of statistical estimation of the parameters of these models was being given serious attention. This activity has grown very rapidly with the introduction of general purpose digital computers of an operating simplicity such that they are readily accessible to a large number of students of economics. Today a large proportion of research in economics is concerned with the formulation and estimation of mathematical models which stem from the early adaptations of classical mechanics.

The other general category of mathematical methods comes under the heading of mathematical programming and game theory. This subject is of relatively recent origin. The first important work in this area was not undertaken until the late 1920's. The mathematical formulations that were given were intended to solve problems which had originally arisen in economics. The development of this category of mathematical analysis in economics was slowed by the fact that new understanding of combinatorial mathematics was required to solve the problems presented by these formulations. Statistical estimation in this area has hardly begun.

2. Prediction in Economics

For the purposes of this paper we may outline the classical development of Wiener (linear) prediction theory for discrete time series as follows. We are given a realization of N equispaced observations of a stochastic process. The process possesses a Cramér representation of the form:

$$x_t = \int_{-\pi}^{\pi} e^{i\omega t} dK(\omega) \tag{2.1}$$

where:

x_t = the time series process at time t,
$K(\omega)$ = probability distribution function.

$K(\omega)$ must have independent increments for x_t to be a linear process, and must have a first derivative which is absolutely continuous on the range $-\pi$ to π in order that the process be purely nondeterministic.

The spectral density function for x_t is of course given by:

$$f(\omega) = k(\omega)k^*(\omega) \tag{2.2}$$

where $k(\omega) = dK(\omega)$, and $k^*(\omega) =$ complex conjugate of $k(\omega)$.

We may then represent x_t by:

$$x_t = \sum_{j=0}^{\infty} k_j \epsilon_{t-j} \tag{2.3}$$

where:

ϵ_t = white Gaussian noise.

The forecast, $\hat{x}_{t+\nu}$, may be represented by:

$$\hat{x}_{t+\nu} = \sum_{j=0}^{\infty} \gamma_j x_{t-j} = \sum_{j=0}^{\infty} \Phi_j \epsilon_{t-j} \tag{2.4}$$

Clearly,

$$\Phi(\omega) = k(\omega)\gamma(\omega) \tag{2.5}$$

where $\Phi(\omega)$ and $\gamma(\omega)$ are the Fourier transforms of Φ_j and γ_j, respectively.

It is usual to measure the error of the forecast by the error variance

$$\begin{aligned} \sigma_\nu^2 &= E[\hat{x}_{t+\nu} - x_{t+\nu}]^2 \\ &= E\left[\sum_{j=0}^{\infty} \Phi_j \epsilon_{t-j} - \sum_{j=0}^{\infty} k_j \epsilon_{t-\nu-j}\right]^2 \\ &= \sigma_\epsilon^2 \left[\sum_{0}^{\nu-1} k_j^2 + \sum_{0}^{\infty} (\Phi_j - k_{j+\nu})^2\right] \end{aligned} \tag{2.6}$$

This last expression is clearly minimized by setting

$$\Phi_j = k_{j+\nu}. \tag{2.7}$$

Thus, if we estimate the parameters k_j we can obtain either the Φ_j's or the γ_j's. Least squares estimates of the k_j may be obtained by estimation of the spectral density function and spectral factorization.

An objection to this procedure was raised in the context of systems theory to the effect that there might exist a feedback from the forecast to the input of the process. Given sufficient information to identify the feedback, this problem has been solved. It is only necessary to recognize that this kind of feedback may exist in many economic applications in order to – in principle – solve the problem.

However, the problem that I want to discuss, while involving an interaction between the forecast and the process, cannot be treated as a feedback problem. In the outline above we have considered a stationary process which is characterized by the function $k(\omega)$. We have, in some sense, assumed that the particular $k(\omega)$ which characterizes the process for which we attempt to forecast future values was randomly chosen from the set of all possible $k(\omega)$. Now we want to consider the consequences for our forecasting procedure of considering the $k(\omega)$ to have been chosen by an opponent whose interest it is to maximize σ_ν^2, subject to certain costs associated with his choices. Thus the opponent controls, within limits, the function $k(\omega)$, while we have under our control the estimation procedure which we will carry out. In particular, we will assume that we have a choice of:

1. the length of the realization (N),
2. the number of parameters which we will attempt to estimate.

The opponent controls $k(\omega)$ by a choice of the coefficients in the z-transform expression for $k(\omega)$:

$$k(z) = \frac{\sum_0^m a_j z^j}{1 - \sum_1^l b_j z^j} \tag{2.8}$$

Since σ_ν^2 is given by:

$$\sigma_\nu^2 = \sigma_\epsilon^2 \Big[\sum_0^{\nu-1} k_j^2 + \sum_0^\infty (\Phi_j - k_{j+\nu})^2 \Big] \tag{2.9}$$

the opponent will attempt to maximize the sum

$$\sum_0^{\nu-1} k_j^2 + \sum_0^\infty (\Phi_j - k_{j+\nu})^2.$$

However, the opponent does not know the estimates that we will supply for the Φ_j and thus his maximization problem depends on our choice of estimation procedure. However, when we attempt to estimate $k(\omega)$ we must implicitly use an assumption concerning the values of l and m in eq. (2.8). We need this information in order to decide on the number of parameters γ_j to estimate, and the length of the realization which we will need.

We will now consider a special case of this model which involves the following further simplifications:

1. The opponent chooses only the parameter l and the values of the parameters b_j of a process with z-transform representation

$$k(z) = \frac{1}{1 - \sum_1^l b_j z^j}. \tag{2.10}$$

Thus the γ_j's which we estimate will be estimates of the b_j's directly. Given a sufficiently long realization from the process with a given l and given values of b_j

we can determine l and estimate the b_j with asymptotically zero error due to the consistency of least squares estimates.

2. The game is played over a long series of consecutive realizations such that we can consider expectations over these realizations.
3. Each realization comes from a given l and set of b_j. The opponent may change l and b_j only between realizations.
4. We have, basically, two choices with respect to estimation of the parameters γ_j. We can estimate a set of γ_j separately for each realization or we can pool realizations.

Clearly, if the opponent changes the parameters in such a way that we can reject the null hypothesis that the parameters are the same, then we will tend to estimate separately. If we cannot reject the null hypothesis, we will tend to combine realizations.

If we now want to look for a reasonable form of optimal behavior for the two participants in this situation we are led naturally to the minimax solution of a 2-person constant sum game. Thus, each player will select that strategy which minimizes the maximum loss which can result from the opponent's possible actions. The payoff of the game is naturally measured in terms of the variance of the forecast error. Thus we have a payoff matrix of the form of Table 1.

Table 1.

		predictor strategies				
		$\gamma_1(z)$	$\gamma_2(z)$	$\gamma_3(z)$	$\cdots$	$\gamma_m(z)$
generator strategies	$k_1(z)$	$\sigma^2_{\nu(1,1)}$	$\sigma^2_{\nu(1,2)}$	$\sigma^2_{\nu(1,3)}$	$\cdots$	$\sigma^2_{\nu(1,m)}$
	$k_2(z)$	$\sigma^2_{\nu(2,1)}$	$\sigma^2_{\nu(2,2)}$	$\sigma^2_{\nu(2,3)}$	$\cdots$	$\sigma^2_{\nu(2,m)}$
	$k_3(z)$	$\sigma^2_{\nu(3,1)}$	$\sigma^2_{\nu(3,2)}$	$\sigma^2_{\nu(3,3)}$	$\cdots$	$\sigma^2_{\nu(3,m)}$
	$\|$	$\|$	$\|$	$\|$	$\|$	$\|$
	$k_n(z)$	$\sigma^2_{\nu(n,1)}$	$\sigma^2_{\nu(n,2)}$	$\sigma^2_{\nu(n,3)}$	$\cdots$	$\sigma^2_{\nu(n,m)}$

In Table 1,

$$\sum \sigma^2_{\nu(p,q)} = \sigma^2_\epsilon \left[\sum_1^{\nu-1} k_j^2(p) + \sum_0^{\infty} (\Phi_j(q) - k_{j+\nu}(p))^2 \right].$$

It is clear that this matrix does not have a saddle point and thus we are led to look for a solution in terms of mixed strategies. We define mixed strategies as:

$$S_g = (\alpha_1 k_1(z), \alpha_2 k_2(z), \alpha_3 k_3(z))$$

$$S_p = (\beta_1 \gamma_1(z), \beta_2 \gamma_2(z), \beta_3 \gamma_3(z))$$

where α_i and β_j are the probabilities of using k_i and γ_j. The payoff function is then:

$$P = \sum_p \sum_q \alpha_p \beta_q \sigma^2_{\nu(p,q)}.$$

The solution is given by:

$$P_o = \min_p \max_q P.$$

Note that since the payoff matrix may be transformed into a Minkowski-Leontief matrix, α_i and $\beta_j > 0$ for all i and j.

3. Conclusion

This paper has attempted to demonstrate one simple point: that for a specified class of models, it is impossible to apply correctly the usual procedures of linear prediction theory. The inferences that one would draw from an application of these techniques would not be correct. We derive a seemingly more reasonable solution to the problem.

The kind of formulation which has been presented may obviously be extended to the problem of estimation of transfer functions given a realization of the input and the output. Extensions such as this are currently being explored.

References

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